Some computational Issues in Nash Equilibria and the Routing Game

Fundamental computational issues concerned with NE

1. **Finding** (efficiently) a mixed/pure (if any) NE
2. Establishing the **quality** of a NE, as compared to a cooperative system, namely a system in which agents can collaborate (recall the *Prisoner's Dilemma*)
3. In a repeated game, establishing **whether** and in **how many** steps the system will eventually **converge** to a NE (recall the *Battle of the Sexes*)
4. **Verifying** that a strategy profile is a NE, **approximating** a NE, NE in **resource** (e.g., time, space, message size) **constrained** settings, **breaking** a NE by colluding, etc…

(interested in a Thesis, or even in a PhD?)

Finding a NE in mixed strategies

- How do we select the correct probability distribution? It looks like a problem in the continuous…

  …but it's not, actually! It can be shown that such a distribution can be found by selecting for each player a best possible subset of pure strategies (so-called best support), over which the probability distribution can actually be found by solving a system of algebraic equations (which are in general exponential in the number of players)

  ⇒ In the practice, the problem can be solved by a simplex-like technique called the *Lemke–Howson algorithm*, which however is exponential in the worst case!

  Remark: Interestingly, 2-player **zero-sum** games can instead be solved in polynomial time!

Is finding a NE NP-hard?

- In pure strategies, **yes**, for many games of interest
- What about mixed strategies? W.l.o.g., we restrict ourselves to **2-player games**, let us call it 2-NASH, **and** we wonder whether 2-NASH (which may be thought in normal form as follows) is NP-hard

	$s_{2,1}$	$s_{2,2}$	...	$s_{2,m}$
$s_{1,1}$	p_1, p_2			
$s_{1,2}$				
. .				
$s_{1,n}$				

Is finding a NE NP-hard? (2)

- **Recall**: a decision problem Π is in NP (resp., in coNP) if all its "yes"-instances (resp., "no"-instances) can be decided in polynomial time by a Non-Deterministic Turing Machine (NDTM) [**Alternative definition** for NP (resp., coNP): set of problems for which a "yes"-instances (resp., a "no"-instances) can be verified in polynomial time by a DTM]
- **Recall also**: a problem Π (not necessarily a decision one) is NP-hard if one can reduce in polynomial time any decision problem Π' in NP to it (this means, Π' can be decided in polynomial time on a NDTM by transforming it to Π, in such a way that "yes"-instances of Π' maps to instances of Π satisfying an easy-to-check predicate, and vice versa)

$\Rightarrow$ It turns out that NP-hardness is then not an appropriate concept of complexity, since we know from Nash's Theorem that every game is guaranteed to have a Nash equilibrium in mixed strategies (i.e., all the instances of 2-NASH are "yes"-instances), and so if 2-NASH would be NP-hard then this would imply that NP = coNP (very hard to believe!)

The complexity class PPAD

- **Definition (Papadimitriou, 1994):** PPAD (*Polynomial Parity Argument – Directed case*) is a subclass of TFNP (Total Function Nondeterministic Polynomial), where existence of a solution is guaranteed by a parity argument. Roughly speaking, PPAD contains all problems whose solution space can be set up as the (non-empty) set of all sinks in a suitable directed graph (generated by the input instance), having an exponential number of vertices in the size of the input, though.
- **Breakthrough:** 2-NASH is PPAD-complete!!!
  (Chen & Deng, FOCS'06)
- **Remark:** It could very well be that PPAD=P≠NP, but several PPAD-complete problems are resisting for decades to poly-time attacks (e.g., finding Brouwer fixed points)

Finding a NE in pure strategies

- By definition, it is easy to see that an entry $(p_1,\dots,p_N)$ of the payoff matrix is a NE if and only if p_i is the maximum ith element of the row $(p_1,\dots,p_{i-1}, \{p(s):s\in S_i\}, p_{i+1},\dots,p_N)$, for each $i=1,\dots,N$.
- Notice that, with N players, an explicit (i.e., in normal-form) representation of the payoff functions is exponential in N $\Rightarrow$ brute-force (i.e., enumerative) search for pure NE is then exponential in the number of players (even if it is still polynomial in the input size, but the normal-form representation needs not be a minimal-space representation of the input!)

$\Rightarrow$ Alternative cheaper methods are sought: for many games of interest, a NE can be found in poly-time w.r.t. to the number of players (e.g., by using the powerful potential method)

On the quality of a NE

- How inefficient is a NE in comparison to an idealized situation in which the players would collaborate selflessly (in other words, the distributed system become **cooperative**), with the common goal of maximizing the overall **social welfare**, i.e., a social-choice function *C* which depends on the payoff of all the players (e.g., *C* is the sum of all the payoffs)?
- **Example**: in the *Prisoner's Dilemma* (PD) game, the DSE (and NE) incurs a total of 10 years in jail for the players. However, if the prisoners would cooperate by not implicating reciprocally, then they would stay a total of only 2 years in jail!

A worst-case perspective: the Price of Anarchy (PoA)

- **Definition (Koutsopias & Papadimitriou, 1999):** Given a game G and a social-choice function C, let S be the set of all NE. If the payoff represents a cost (resp., a utility) for a player, let OPT be the outcome of G minimizing (resp., maximizing) C. Then, the *Price of Anarchy* (PoA) of G w.r.t. C is

$$\mathrm{PoA}_G(C) = \sup_{s \in S} \frac{C(s)}{C(\mathrm{OPT})} \left(\text{resp.}, \inf_{s \in S} \frac{C(s)}{C(\mathrm{OPT})} \right)$$

- **Example**: in the PD game, $\mathrm{PoA}_{PD}(C)=10/2=5$

A case study for the existence and quality of a NE: selfish routing on Internet

- Internet components are made up of heterogeneous nodes and links, and the network architecture is open-based and dynamic
- Internet users behave selfishly: they generate traffic, and their only goal is to download/upload data as fast as possible!
- But the more a link is used, the more is slower, and there is no central authority "optimizing" the data flow…
- So, why does Internet eventually work is such a **jungle**???

The Internet routing game

Internet can be modelled by using game theory: it is a (**congestion**) **game** in which

players	$\longrightarrow$	users
strategies	$\longrightarrow$	paths over which users can route their traffic

Non-atomic Selfish Routing:

- There is a large number of (selfish) users generating a large amount of traffic;
- Every user controls an infinitesimal fraction of the traffic;
- The traffic of a user is routed over a single path in one shot.

Mathematical model (multicommodity flow network)

- A directed graph $G = (V,E)$ and a set of N players
- A set of *commodities*, i.e., source-sink pairs (s_i,t_i), for $i=1,..,k$ (each of the $N\geq k$ players is associated with a commodity)
- Let N_i be the amount of players associated with (s_i,t_i), for each $i=1,..,k$; then, the rate of traffic between s_i and t_i is $r_i=N_i/N$, with $0\leq r_i \leq 1$ and $\Sigma_{i=1,\dots,k}\, r_i = 1$
- A set Π_i of paths in G between s_i and t_i for each $i=1,..,k$, and the corresponding set of all paths $\Pi=\cup_{i=1,\dots,k}\,\Pi_i$
- **Strategy for a player**: a path joining its commodity
- **Strategy profile**: a flow vector f specifying the rate of traffic f_P routed on each path $P\in\Pi$ (notice that $0\leq f_P \leq 1$, and that for every $i=1,..,k$ we have $\Sigma_{P\in\Pi_i} f_P = r_i$)

Mathematical model (2)

- For each $e \in E$, the amount of flow absorbed by e w.r.t. f is $f_e = \Sigma_{P \in \Pi : e \in P} f_P$
- For each edge e, a real-value latency function $l_e(x):[0,1] \rightarrow \Re^+$ of its absorbed flow x (this is a monotonically non-decreasing function which expresses how e gets congested when a fraction $0 \leq x \leq 1$ of the total flow f uses e)
- Cost of a player: the latency of its used path $P \in \Pi$:

$$c(P) = \Sigma_{e \in P} \; l_e(f_e)$$

- Cost (or average latency) of a flow f (**social-choice function**): $C(f) = \Sigma_{P \in \Pi} f_P \cdot c(P) = \Sigma_{P \in \Pi} f_P \cdot \Sigma_{e \in P} l_e(f_e) = \Sigma_{e \in E} f_e \cdot l_e(f_e)$

Observation: Notice that the game is not given in normal form!

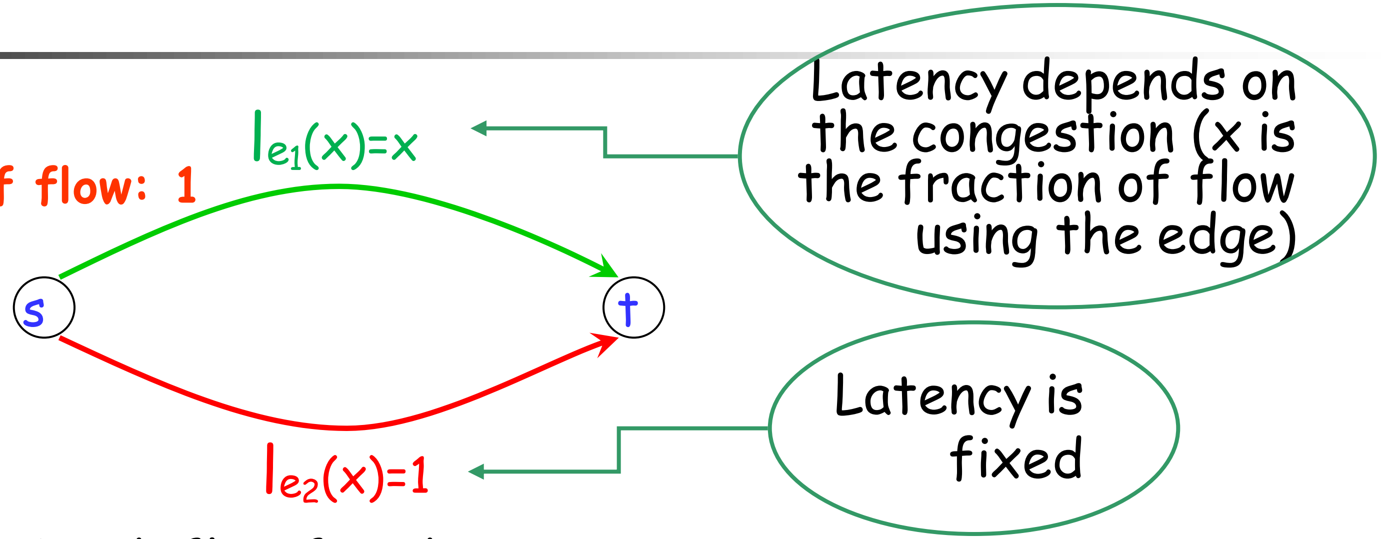
Flows and NE

Definition: A flow f^* is a Nash flow if no player can improve its cost (i.e., the cost of its used path) by changing unilaterally its path.

QUESTION: Given an instance $(G,s=((s_1,t_1),\ldots,(s_k,t_k)),r=(r_1,\ldots,r_k),l=(l_{e_1},\ldots,l_{e_m}))$ of the non-atomic selfish routing game, does it admit one or more Nash flows? And in the positive case, what is the PoA of the game?

Example: Pigou's game [1920]

Total amount of flow: 1



- Is there any Nash flow for this game?
  YES! For instance, that in which all the flow travels on the upper edge
  $\Rightarrow$ the cost of this flow is $C(f) = 1 \cdot l_{e_1}(1) + 0 \cdot l_{e_2}(0) = 1 \cdot 1 + 0 \cdot 1 = 1$
- Are there any other Nash flows? NO
- What is the PoA of this game? The optimal solution is the minimum of $C(x)=x \cdot x + (1-x) \cdot 1 \Rightarrow C(x)=x^2-x+1 \Rightarrow C'(x)=2x-1 \Rightarrow$ OPT for $C'(x)=0$, i.e., $x=1/2 \Rightarrow C(OPT)=1/2 \cdot 1/2+(1-1/2) \cdot 1=0.75$

$\Rightarrow$ PoA(C) = 1/0.75 = **4/3**

Existence of a Nash flow

- **Theorem (Beckmann *et al.*, 1956):** If for each edge e the function $x \cdot l_e(x)$ is convex (i.e., its graphic lies below the line segment joining any two points of the graphic) and continuously differentiable (i.e., its derivative exists at each point in its domain and is continuous), then the Nash flow of (G,s,r,l) exists and is unique, and is equal to the optimal min-cost flow of the following instance:

$$(G,s,r,\lambda(x)=[\textstyle\int_0^x l(t)dt]/x).$$

- **Remark**: The optimal min-cost flow can be computed in polynomial time through convex programming methods.

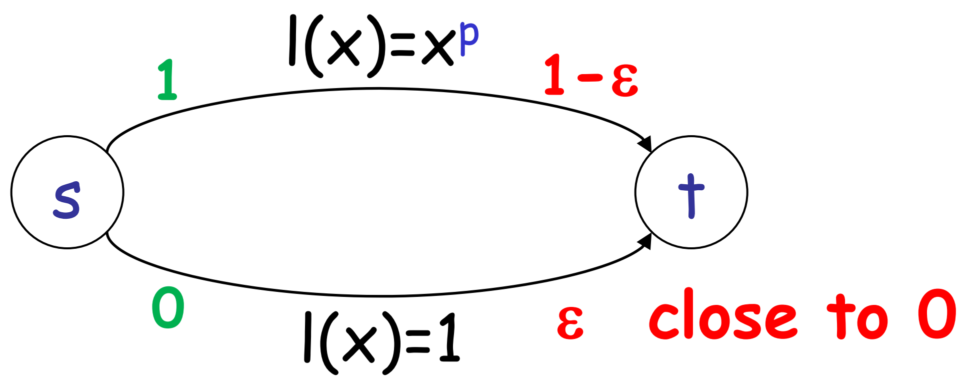
Flows and Price of Anarchy

- Theorem 1: In a network with linear latency functions, the cost of a Nash flow is at most 4/3 times that of the min-cost flow $\Rightarrow$ every instance of the non-atomic selfish routing satisfying this constraint has PoA $\leq$ 4/3.
- Theorem 2: In a network with degree-p polynomials latency functions, the cost of a Nash flow is $O(p/\log p)$ times that of the min-cost flow.

(Roughgarden & Tardos, JACM'02)

A bad example for non-linear latencies

Assume $p \gg 1$

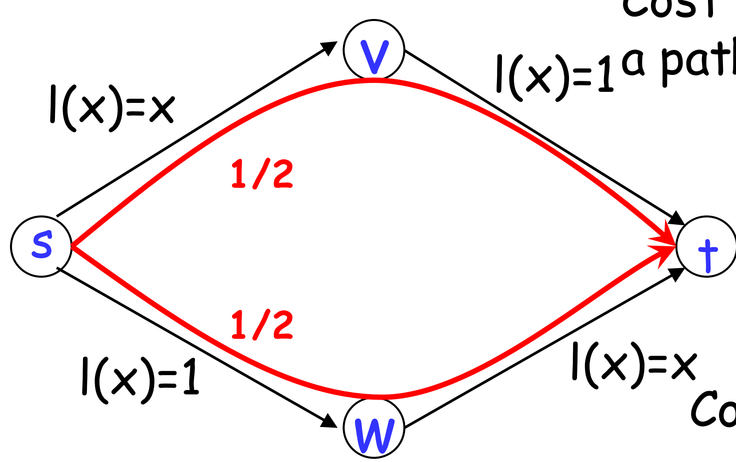


A **Nash flow** (of cost $C=1 \cdot 1^p+0 \cdot 1=1$) is arbitrarily more expensive than the **optimal flow** (of cost $C=(1-\varepsilon) \cdot (1-\varepsilon)^p+ \varepsilon \cdot 1 \approx 0$)

Improving the PoA: the Braess's paradox

- Does it help adding edges to improve the PoA?
- **NO!** Let's have a look at the **Braess Paradox** (1968)

Cost for each player (i.e., latency of a path) = x+1=**1/2**+1 = 1.5

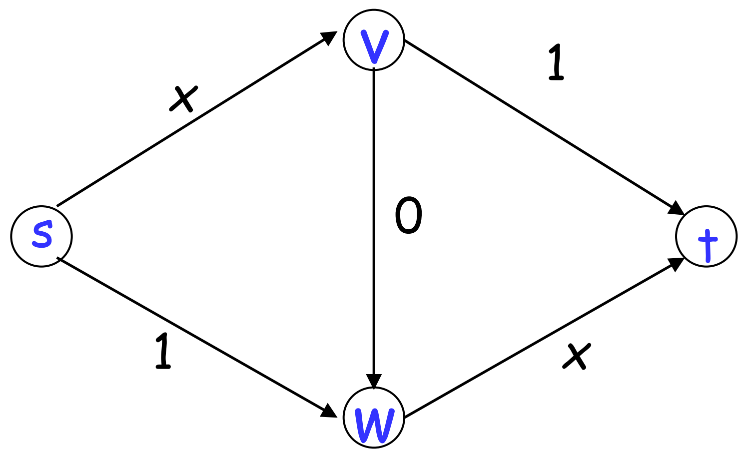


Cost of the flow= 2·(1.5·**1/2**)=1.5

(notice this a NE and it is also an **optimal flow**)

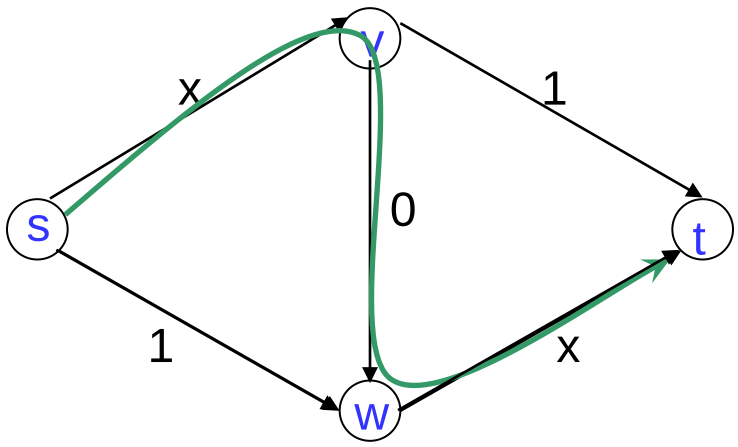
The Braess's paradox (2)

To reduce the cost of the flow, we try to add a no-latency road between v and w. Intuitively, this should not worse things!



The Braess's paradox (3)

However, each user is tempted to change its route now, since the path $s \to v \to w \to t$ has less cost (indeed, $x \leq 1$)



If only a single user changes its route, then its cost decreases from 1.5 to approximately 1, i.e.:

$c(s \to v \to w \to t) = x+0+x \approx 0.5 + 0.5 = 1$

But the problem is that all the users will decide to change!

The Braess's paradox (4)

- So, the cost of the flow f that now entirely uses the path $s\rightarrow v\rightarrow w\rightarrow t$ is:

$$C(f) = 1\cdot 1+1\cdot 0+1\cdot 1=2>1.5$$

- Even worse, this is a NE (the cost of the path $s\rightarrow v\rightarrow w\rightarrow t$ is 2, and the cost of the two paths not using (v,w) is also 2)!
- The optimal min-cost flow is equal to that we had before adding the new road and so, the PoA is

$$PoA = \frac{2}{1.5} = \boxed{\frac{4}{3}}$$

Notice it is 4/3, as in the Pigou's example, and it is equal to the upper bound we gave for linear latency functions

Convergence towards a NE (in pure strategies games)

- Ok, we know that selfish routing is not so bad at its NE, but are we really sure this point of equilibrium will be eventually reached?
- **Convergence Time**: number of moves made by the players to reach a NE from an initial arbitrary state
- **Question:** *Is the convergence time (polynomially) bounded in the number of players?*

Convergence towards the Nash flow

- ☺ Positive result: If players obey to a *best response dynamics* (i.e., each player at each step greedily selects a strategy which maximizes its personal utility) then the non-atomic selfish routing game will converge to a NE. Moreover, for many instances (i.e., for prominent graph topologies and/or commodity specifications), the convergence time is polynomial.
- ☹ Negative result: However, there exist instances of the non-atomic selfish routing game for which the convergence time is exponential (under some mild assumptions).